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ON EDDY DIFFUSIVITY, QUASI-SIMILARITY AND DIFFUSION EXPERIMENTS IN TURBULENT BOUNDARY LAYERS

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Abstract—The experimental evidence of Pöreh and Cermak [1] of two zones of quasi-similarity in a plume diffusing from a line source at the wall of a developed turbulent layer is carefully examined for consistency with the concepts of eddy diffusivity. The quasi-similarity of mean temperature profiles within a developed turbulent layer some distance downstream from a step in wall temperature observed by Johnson [2a], [2c], is similarly analysed. It is concluded that eddy diffusivities when viewed as properties of quasi-similar fields can account for the observed characteristics of the layer developing within another layer to the accuracy of observations. Some consequences of these concepts are explored.

NOMENCLATURE

c, instantaneous increment in concentration from the mean;
c_f, local skin friction coefficient;
C, concentration;
C_p, specific heat at constant pressure;
d, one of the several thickness of the boundary layer;
f, dimensionless concentration, equation (5_C);
g, dimensionless temperature, equation (5_t);
h, second approximation to dimensionless velocity;
H, dimensionless stream function;
i, integral defined in equation (12_t);
I, integral defined in equation (8_C);
k, heat conductivity;
l, some measure of plume width;
L, length of turbulent boundary layer upstream of the origin;
n, inverse of the exponent in the power-law approximation of velocity profile, equation (11);
Pr, Prandtl number;
Pr_t, turbulent Prandtl number, ϵ_u/ϵ_t;
q, heat flux per unit area in y direction;
Re, Reynolds number;
R_{ut}; correlation coefficient, $\overline{ut}/\sqrt{(\overline{u^2})}\sqrt{(\overline{t^2})}$;
St, Stanton number, $q_w/\rho C_p U_e \Delta T$;
T, mean temperature;
ΔT, $T_w - T_e$;
t, instantaneous temperature fluctuation from the mean;
u, v, instantaneous velocity fluctuation components in the horizontal stream direction and the direction perpendicular to the wall;
u_*, Prandtl's friction velocity, $\sqrt{(\tau_w/\rho)}$;
U, mean velocity component in the stream direction;
V, mean velocity component perpendicular to the wall;
x, distance downstream from the line source of ammonia or from the leading edge of heated plate;
y, distance from the wall;
y^*, dimensionless distance, yu_*/ν;
α, shape indicator of concentration profiles $\mathrm{d}\ln C_w/\mathrm{d}\ln\lambda$;
β, measure of relative development of diffusive plumes and of boundary layer, $\mathrm{d}\ln d/\mathrm{d}\ln\lambda$, $d_x\lambda/d\lambda_x$;
γ, $\mathrm{d}\ln I_c/\mathrm{d}\ln\lambda$, equation (9);
δ, boundary layer thickness based on $U/U_e = 0{\cdot}99$, or equivalent;
δ_L, laminar sublayer;
δ^*, displacement thickness;

Δ, "Clauser thickness", approximately $3{\cdot}6\delta$;
ϵ, eddy-transport coefficient, equation (1);
ζ, y/θ;
η, y/δ or y/d;
θ, momentum thickness;
λ, diffusive plume scale, standardized so that at $y = \lambda$, 50 per cent of change from wall to free stream values is effected;
μ, viscosity;
ν, kinematic viscosity, μ/ρ;
ξ, y/λ or y/l;
ρ, density;
τ, shear stress;
τ_{2w}, τ_w/ϵ_{u1}.

Subscripts
$()_B$, pertaining to outer edge of buffer layer;
$()_C$, pertaining to (mass) concentration field;
$()_e$, at edge of boundary layer, i.e. free stream values;
$()_L$, pertaining to edge of laminar sublayer;
$()_t$, pertaining to temperature field;
$()_u$, pertaining to velocity field;
$()_w$, wall value;
$()_x$, partial derivative with respect to x.

Superscripts
$\overline{()}$, time average;
$()'$, derivative with respect to ξ;
$()^*$, derivative with respect to η.

INTRODUCTION

THE PAPER, "Study of diffusion from a line source in a turbulent boundary layer", by M. Poreh and J. E. Cermak [1] is particularly interesting because it documents two regions of quasi-similarity, designated as an intermediate zone and a final zone, in the growth of a diffusive field within an already well developed turbulent boundary layer [see Fig. 1(a)]. Another intriguing result is the apparent lack of dependence of the vertical scale of the diffusing plume on the free stream velocity.

The final zone of quasi-similarity of [1] occurs when the diffusing ammonia gas has effectively spread over the whole turbulent layer so that the characteristic scales of the vortical and the ammonia concentration layers become commensurate and can be expected to develop hand in hand. The Poreh–Cermak measure of the relative development of these scales, $\beta = d \ln d/d \ln l$, which otherwise is a function of distance,† $\beta(x)$, then approaches unity. The intermediate zone of quasi-similarity corresponds to a region closer to the source, where the plume grows rapidly in a relatively slowly changing vortical layer so that β is relatively small, ranging from 0·08 to 0·30 according to the experiments [1].

The wall being impermeable to the gas, the boundary conditions for this Dirac-like source problem differ‡ from those encountered in problems of heat diffusion from a wall with a Heaviside-like step in wall temperature (Johnson [2], Reynolds, Kays, and Kline [3]). Still, both fields are scalar fields convected and contorted by a pre-existent slowly developing turbulent field and are characterized by two scales, the individual plume widths, and the strengths of the gas source and of the temperature step, respectively. Johnson [2a, 2c], who also had ample room for observation in his thick boundary layer, indeed noted the strong similarity of mean profiles between 25 to 30 and 60 to 65 in downstream from the step (see Fig. 2). The region corresponds to approximate $d \ln \theta/d \ln \lambda$ values from 0·12 to 0·27, i.e. to an "intermediate zone". Unfortunately, Johnson's hot plate was not long enough to indicate the shift toward the expected final zone of quasi-similarity, sometimes referred to as "isothermal" (Reynolds, Kays, and Kline [7], Elias [9]). The shift from Johnson's profiles to the isothermal-plate profiles indicated in Fig. 2, is substantial and undoubtedly significant.

The present discussion is concerned with the consistency between these findings and concepts of eddy diffusivity and of the turbulent Schmidt or Prandtl number. The history of the turbulent field and of the diffusive field being strongly dissimilar, these concepts are subjected here to a particularly stringent and challenging test. In the analysis, the quasi-similarities will be postulated as empirical facts. However, the very use of the

† Approximately, $\beta = x/(x + L)$.
‡ In particular, in the ammonia case one cannot have recourse, without considerable sophistication, to concepts of local Reynolds analogy between the velocity and the scalar field near the wall.

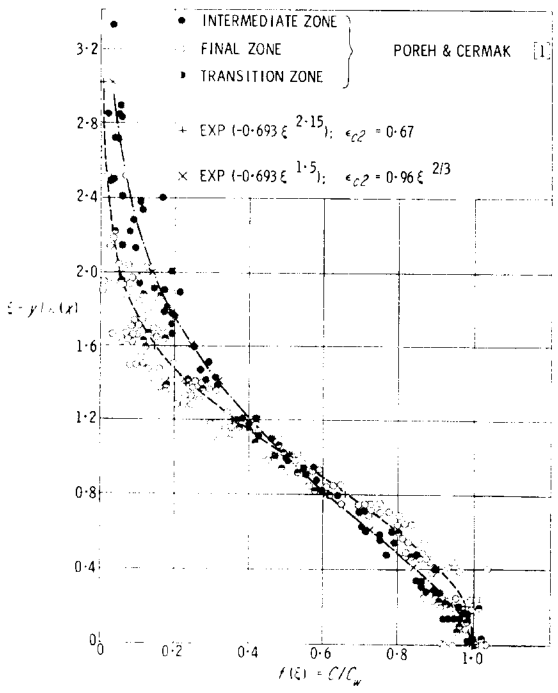

FIG. 1(a). Comparison of experimental and computed concentration profiles.

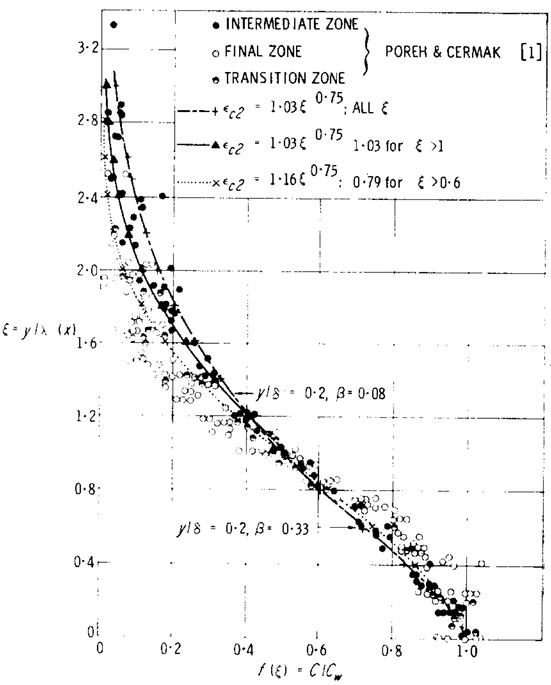

FIG. 1(b). Comparison of experimental and computed concentration profiles.

prefix "quasi" is intended to emphasize that these similarities are no more "true" than that of the turbulent layer itself, a layer which possesses an inertial scale and a viscous scale developing at different rates. It will in fact be seen that the postulated similarity for the more sensitive wall-step case requires a decrease of eddy conductivity from the inertially controlled turbulent level in the middle of the layer to the laminar level near the wall if the shape of the Johnson profiles of Fig. 2 is to be well approximated. The impermeability of the wall in the ammonia field decreases the sensitivity of the overall diffusion process to the viscous scale near the wall, making the inclusion of the "laminar sublayer" unnecessary. Incidentally, throughout the paper the traditional terminology of the laminar sublayer and of the buffer layer has been retained to indicate simply the wall regions where the molecular diffusivities dominate or are comparable to local eddy diffusivities. The molecularly influenced domain of the law of the wall extends considerably farther outward. The turbulent structure in this domain is considered in the closing section of the paper.

EDDY TRANSPORT AND TURBULENT PRANDTL NUMBER

Concepts of eddy transport coefficients have been assessed critically by Corrsin, Townsend, Batchelor, and others, especially for free turbulent flows, and excellent reviews of the many facets of the problem were made readily available by Hinze [4] and Schubauer and Tchen [5]. For monotonic velocity–temperature fields the hot-wire anemometer provides a direct means of evaluating the Boussinesq eddy coefficients locally from the defining relationships:

$$\epsilon_u = -\overline{uv}\bigg/\frac{\partial U}{\partial y};\ \epsilon_t = -\overline{tv}\bigg/\frac{\partial T}{\partial y} \qquad (1)$$

The uncertainties in such determination of the diffusivities run on the order of ± 10 per cent in very careful experiments and are somewhat reduced in the determination of the ratio $\epsilon_u/\epsilon_t = Pr_t$, the turbulent Prandtl number,

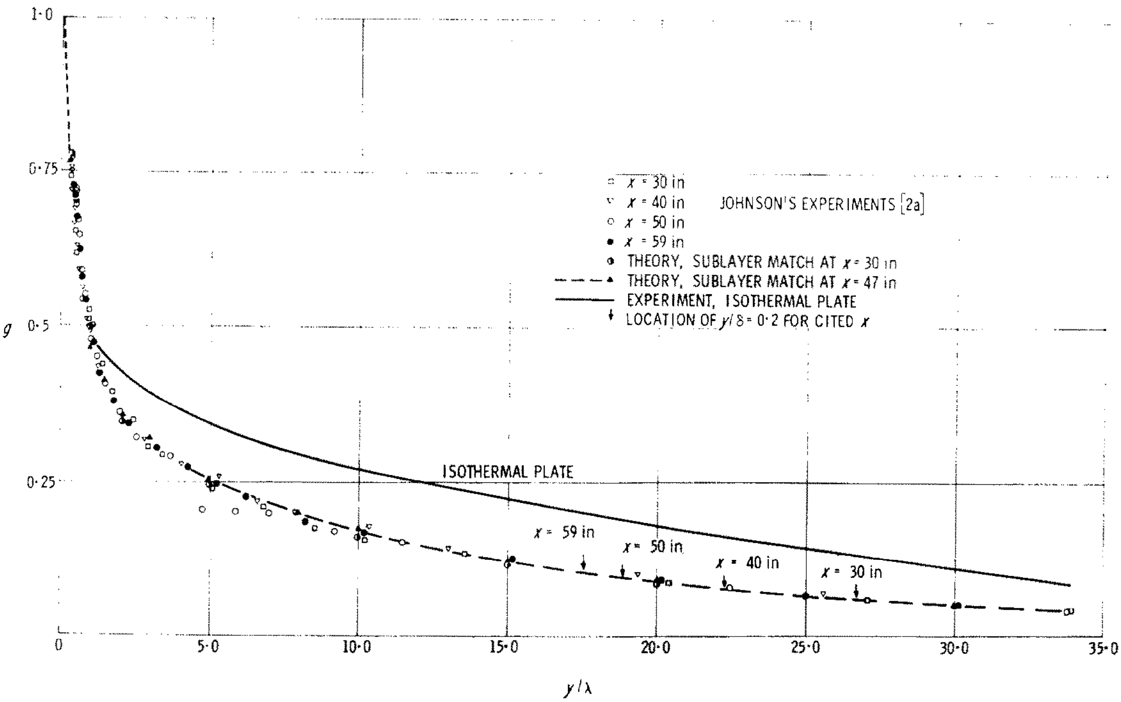

FIG. 2. Comparison of experimental and computed temperature profiles.

because in forming the ratio some systematic errors cancel. Thus, in the interesting region, specifically, at $x = 47$ in, corresponding to the intermediate zone of Poreh and Cermak, Johnson [2] reports Pr_t rising from 0·75 at the edge of the Karman buffer layer near the wall to a plateau of 1·08 within the warm ($\Delta T_{\max} = 15°C$) plume and starting to decline near $y/\theta = 1{\cdot}3$, still inside the plume, before a sharp temperature intermittency appears in the middle of the turbulent velocity region ($0{\cdot}27 < y/\delta < 0{\cdot}65$) and the smallness of the signals begins to impair the accuracy of the results. For velocity and temperature fields developed concurrently in presence of invariant wall conditions, Pr_t obtained by the hot-wire technique usually exhibits the rise near the wall and a plateau between 0·8 and 0·9 before the intermittent bulges of the outer layer are reached.† Should the difference between the 1·08 plateau in the narrow plume of Johnson and the common 0·8 to 0·9 plateau be significant, the eddy coefficient ϵ_t would exhibit dependence on the specific temperature field, which ought to have been erased in dividing $-\overline{tv}$ by $\partial T/\partial y$ in equation (1) in the case of a gradient transport controlled by the local kinematic field.

The striking temperature intermittency observed by Johnson ([2b], Fig. 6) in the midst of a layer where there is no velocity intermittency (a basic finding which has not been accorded sufficient attention) testifies to the presence of contributions to overall transport by the large eddies, not generally considered consistent with the ideas of gradient transport. However, it has been pointed out by Liepmann [10], that while the eddy-coefficient concept originated from a gradient-diffusion model, the resulting expression [i.e. equation (1)] does not necessarily imply the specific mechanism. He calls attention to the fact that "process of collisionless diffusion of clouds into vacuum can be rigorously interpreted as a diffusion process of the gradient type with a diffusion coefficient proportional to

† In turbulent jets and wakes somewhat smaller Pr_t, 0·6 to 0.7, are usually observed by the same techniques. The difference may be symptomatic of the deeper bulges (more widely spread intermittency) of free turbulent flows, ([5], p. 165), and of the more important role of the large eddies.

time, i.e. with precisely the same form of expression as used in Prandtl's second formulation of the free turbulent exchange."

In equation (1), attention is focused on the transport of the incremental heat per unit volume, $C_p t$, and of the incremental horizontal momentum per unit volume, u, by the lateral velocity fluctuations, v. A hint of the mechanisms of transport can also be obtained from the correlation coefficient between the two transported properties, namely $R_{ut} = \overline{ut}/\sqrt{(\overline{u^2})}.\sqrt{(\overline{t^2})}$ which is more easily measureable (i.e. with a single hot wire). Johnson's Fig. 5 shows that R_{ut} is negative and decreases steadily in absolute value with distance from the wall from about 0·78 to 0·6 before the temperature intermittency sets in. By taking spectra of R_{ut} in the case of a high-speed insulated boundary layer, Morkovin ([11], Fig. 5) showed that the correlation can exceed 0·9 for the largest eddies, that it decreases with eddy size, and that one could expect the correlation to be maintained across the whole layer in the low-speed cases of constant wall temperature, Such information points again to the transport role of the larger eddies.

It will be seen that adoption of the hypothesis of equation (1) in the present case will lead to a generalization of Prandtl's formulation referred to by Liepmann. Liepmann's comment [10] to the effect that he now feels that the phenomenological approaches, if "recast and reinterpreted in relation to recent results both in turbulence and in the theory of fluids, still have a future," is also pertinent to the present difficult case of a layer within a layer.

EDDY DIFFUSIVITY AS PROPERTY OF QUASI-SIMILAR FIELDS

The present writer feels that the role of the large eddies has by now been so well documented (Townsend, Corrsin, Laufer, Klebanoff, Favre, Grant) that it is not particularly fruitful to attempt to visualize the details of the mechanism of interplay between the large eddies and the smaller ones in setting up formulae for eddy coefficients in the many cases where they appear to work. Rather one could expect that after a sufficient time (distance) of a sufficiently uniform development of the fields in question, the effects of the large eddies, in conjunction with smaller ones, would be also statistically smoothed out and a state of quasi-equilibrium or quasi-similarity reached, in which the average transport could behave as if due to eddy diffusivities, with expected departures in the proximity of walls and in the regions of intermittency. If so, existence of local quasi-similarity of mean profiles suggests sufficient uniformity of locally dominant average behavior and hence the possible usefulness of eddy coefficients and their possible relations to the usually non-constant characteristic scales of the developing fields.

It was the similarity afforded by the velocity defect law in otherwise non-self-preserving turbulent boundary layers that prompted Clauser [6] to document the applicability of the eddy viscosity concept to the outer 80 to 90 per cent of layers with constant pressure and with equilibrium pressure gradients. The useful, though rather neglected, result, namely

$$\epsilon_u(x) = 0{\cdot}018\, U_e\delta^* = 0{\cdot}018\, u_*\Delta \qquad (2)$$

displays the dependence upon the local characteristic scales U_e and δ^* observed from the outer "potential" flow on one hand, and on the combination of an inner scale, u_*, and an overall inertial scale Δ, on the other. Encouraged by this success, let us pursue the possibility that the quasi-similarities observed by Poreh and Cermak and by Johnson may lead to useful eddy coefficients, within the accuracy of the observations.

At the outset, we should not expect to obtain results comparable in their validity to that of equation (2) because, as Poreh and Cermak observe, their similarities are at best asymptotic, and because in the intermediate zone the influence of the viscous sublayer could remain strong. In fact, this influence vitiates the asymptotic similarity† of the initial zone, which is present in comparable experiments in free turbulent

† There could possibly be a hidden initial-zone similarity based on molecular parameters similar to u_* and y^* of the law of the wall. A hint of such a similarity exists in Johnson's measurements for $x \leqslant 20$ in, but uncertainty in the local values of the Stanton number and skin friction coefficient precludes any definite conclusions. There was also scattered evidence of changes of the viscous layer thickness due to heating, even of possible buoyancy effects in this region of low velocity.

flows, e.g. Hinze ([4], p. 351). The region near the wall also exhibits larger scatter of the "carrier" velocity field—see Poreh and Cermak's Fig. 2, where many of the profiles in the intermediate zone could be more closely approximated by 1/6 power law. We also observe larger departures of the individual profiles from the "mean" similarity profile in Figs. 5 and 10 of Poreh and Cermak [1], or in our Fig. 1, in the regions near the wall and near the edge of the plume, and we should expect the eddy-diffusivity concept to lead to no better consistency. Similar statements hold for the velocity and temperature profiles of Johnson, e.g. our Fig. 2. In particular the experimental profiles of Poreh–Cermak appear to approach the wall with a non-zero slope, not appropriate to impermeable walls, and we can expect a sharp curvature of the profiles at the edge of the viscous sublayer, reminiscent of the bends in the temperature profiles at insulated walls (leading to recovery factors near 0·9 in air).

SOME CONSEQUENCES OF QUASI-SIMILARITY

The flow being incompressible, there exists a mean stream function $\psi(x, y)$, which in the inertially controlled region of similarity of a turbulent boundary layer should be expressible in terms of the characteristic scales U_e and d (an appropriate thickness):

$$\psi(x, y) = d(x)\, H\left(\frac{y}{d}\right); \ \frac{y}{d(x)} = \eta \qquad (3)$$

so that

$$\frac{U(x, y)}{U_e} = H^*(\eta), \frac{V(x, y)}{U_e} = -\, d_x\,[H - \eta H^*] \quad (4)$$

where the subscript x and superscript ( )* denote differentiation with respect to x and η, respectively. In regions where the diffusive field is self-similar, Poreh and Cermak expressed the concentration profiles as

$$C(x, y) = C_w(x) f(\xi); \ \xi = \frac{y}{l(x)} \qquad (5\text{c})$$

where further specification of the vertical scale of the plume, $l(x)$, may be delayed so that the equations remain applicable to both the intermediate and final zones. Using the Poreh–Cermak indicator of the relative growth of the velocity and the diffusive boundary layers, $\beta = \mathrm{d}(\ln d)/\mathrm{d}(\ln l)$, and introducing a shape indicator of the diffusive profiles, $\alpha = \mathrm{d}(\ln C_w)/\mathrm{d}(\ln l)$, the differential equation for the conservation of mass of ammonia,

$$U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} = \frac{\partial (UC)}{\partial x} + \frac{\partial (VC)}{\partial y} = \frac{\partial}{\partial y}\left(k\,\frac{\partial C}{\partial y} - \overline{vc}\right) \quad (6\text{c})$$

specializes to

$$\left.\begin{aligned} & -\,U_e\,C_w\,\frac{l_x}{l}\Big[-\,\alpha f H^* + \xi f'\,H^* + \\ & \beta\eta\,H^{**} f + \frac{\partial}{\partial y}\,\{d\beta f H - d\beta\eta\,H^* f\,\}\Big] = \\ & \frac{\partial}{\partial y}\left[k\,C_w\,\frac{\partial f}{\partial y} - \overline{vc}\right] \end{aligned}\right\} \quad (7\text{c})$$

When we assume that the similarity is valid across the whole layer† (including the sublayer), and integrate equation (7c) across it, applying the conditions of impermeability at the wall, and then integrate with respect to x, we obtain the overall conservation condition equation (12) of Poreh and Cermak:

$$U_e\,C_w(x)\;l(x)\,I_c(x) = G = \text{const};$$

$$I_c(x) = \int_0^\infty H^*\left(\xi\,\frac{l}{d}\right) f(\xi)\,\mathrm{d}\xi \qquad (8\text{c})$$

The flux integral $I_c(x)$ displays the basic mathematical difficulty in treating fields with two distinct similarity variables. Nevertheless, equation (8c), logarithmically redifferentiated, provides a simple basic constraint on the rate of evolution of the concentration field:

$$1 + \alpha + \gamma = 0; \ \gamma = \frac{\mathrm{d}\ln I_c}{\mathrm{d}\ln l} \qquad (9)$$

† For the impermeable wall, the assumption of similarity at the wall is actually mild, because the possible non-similar contribution to the growth of the function $f(\xi)$ in the viscous layer $\left(\frac{\partial f}{\partial y}\right)_w y + \left(\frac{\partial^2 f}{\partial y^2}\right)_w y^2/2 + \ldots$, must be very small since $\left(\frac{\partial f}{\partial y}\right)_w$ vanishes according to the boundary condition and $\left(\frac{\partial^2 f}{\partial y^2}\right)_w$ according to left side of equation (6c), i.e. as U and V approach zero.

When x measures the distance from the line source of ammonia, the experimental results of Poreh and Cermak indicate that $C_w \sim x^{-0\cdot 9}\pm$, $l \sim x^{0\cdot 8}\pm$, and hence $\alpha = -\frac{9}{8}$, $\gamma = \frac{1}{8}$, in the intermediate zone, and $C_w\, l \sim x^0$, $\alpha = -1$, $\gamma = 0$, in the final zone. The relative influence of the dependence on x of the flux integral I_c is thus seen to be slight, corresponding to a change of γ from $\frac{1}{8}$ to zero during the part of the development of the diffusion layer under scrutiny.

The nearly self-similar velocity field is governed by a single differential equation of conservation of horizontal momentum, parallel in content to equation (7c)

$$-U_e^2 \frac{d_x}{d} HH^{**} = \frac{\partial}{\partial y}\left(\nu \frac{\partial H^*}{\partial y} - \overline{uv}\right) \approx + \frac{U_e}{d^2}[\epsilon_u H^{**}]^* \qquad (10)$$

Recognizing that the similarity equation (3) was to be expected only in the outer inertial region of the boundary layer, Clauser [6] solved equation (10) with judicious handling of the inner boundary conditions, approximating those seen by the fully turbulent fluid, derived equation (2), and documented the sense in which the Boussinesq approximation, indicated on the right of equation (10), could be considered valid in turbulent boundary layers with equilibrium pressure gradients.

An approximation to these profiles is given by the power law profiles

$$\frac{U}{U_e} = H^* = \eta^{1/n};\ \psi = U_e\, d(x)\, \frac{n}{n+1}\, \eta^{(n+1)/n} \qquad (11)$$

It should be understood that this approximation† conceals a slow dependence on x (or Re_x) in n, and gives a deceptive impression of possessing full similarity in the wall region and of satisfying the boundary conditions at the wall, where it yields an infinite shear. Nevertheless, with judicious handling of these inner conditions, they can be very useful, even though they lead to occasional inconsistencies because of their overdeterminacy. Poreh and Cermak recognized that the power-law is the functional form that can overcome the mathematical difficulties caused by the double similarity in the present problem. Thus, in our formulation the integral $I_c(x)$ becomes separable and equations (8c) and (9c) become

$$I_c(x) = \left(\frac{l(x)}{d(x)}\right)^{1/n} \int_0^\infty \xi^{1/n} f(\xi)\, d\xi;\ \gamma = \frac{1}{n}(1-\beta) \qquad (12c)$$

This approximation thus predicts a shift in γ from $\frac{1}{6}$ or $\frac{1}{7}$ in the intermediate zone (β small), to zero in the final zone ($\beta = 1$), as compared to the experimentally indicated shift from $\frac{1}{8}$ to zero. This agreement is nearly within the experimental accuracy, a new and encouraging result.

Returning to the basic differential equation (7c), we must make an additional assumption relating the turbulent mass flux $\overline{vc}$ to the mean fields of C and U. We explore the feasibility of using the eddy coefficient, ϵ_C, which could depend on both x and ξ, namely

$$-\overline{vc} = \epsilon_C\,(x, \xi)\frac{\partial C}{\partial y} = \epsilon_C\,(x, \xi)\,\frac{C_w(x)}{l(x)}\, f'(\xi) \qquad (13)$$

and, neglecting again the small viscous contributions near the wall, we obtain:

$$-U_e\, l_x\, d(x)\, f(\xi)\, \eta^{(n+1)/n}\left(1 - \frac{\beta(x)}{n+1}\right) = \epsilon_C(x, \xi)\, f'(\xi) \qquad (14c)$$

Before proceeding further, it is instructive to consider the differences brought about by the changes of boundary conditions from those of Poreh–Cermak to those of Johnson. The corresponding equations are designated by the subscripts C and t, respectively. For the tempera-

† Note that the only power allowed by the strict similarity equation (11) for the velocity profiles is -1, and yet we know how useful powers like $\frac{1}{5}$ to $\frac{1}{7}$ can be, with proper interpretation; see Hinze [4], pp. 481 to 482). Note also, that the approximating form enters essentially as a weighting function, the result to be integrated, i.e. smoothed. In such an operation the overall shape rather than the local values are important.

ture field with a fixed step ΔT in the wall temperature at $x = 0$, they are:

$$T(x, y) - T_e = \Delta T\, g(\xi);\ \xi = \frac{y}{l(x)} \qquad (5_t)$$

$$\frac{\partial(UT)}{\partial x} + \frac{\partial(VT)}{\partial y} = \frac{\partial}{\partial y}\left[\frac{k}{\rho\, C_p}\frac{\partial T}{\partial y} - \overline{vt}\right] = -\frac{\partial q}{\partial y} \qquad (6_t)$$

$$\frac{\partial}{\partial x}\,[l(x)\, I_t(x)] = \frac{q_w(x)}{\rho\, C_p\, U_e\, \Delta T} = St(x);$$

$$I_t(x) = \int_0^\infty H^*\left(\xi\frac{l}{d}\right) g(\xi)\, \mathrm{d}\xi \qquad (8_t)$$

The term in the bracket of equation (8_t) has the dimensions of a length and has been called the convection thickness of a thermal boundary layer by Eckert [12]. It is proportional† to the plume scale $l(x)$ only in the final zone where I_t approaches a constant as l/d approaches a constant.

The permeability of the wall to heat brings forth the non-vanishing value of the local Stanton number, and equations (9) and (12) are replaced by

$$l_x\left(\frac{l}{d}\right)^{1/n} i_t \frac{n+1}{n}\left(1 - \frac{\beta}{n+1}\right) = St(x);$$

$$i_t = \int_0^\infty \xi^{1/n}\, g(\xi)\, \mathrm{d}\xi = \text{const.} \qquad (12_t)$$

The experimental check here is of necessity less positive than in the ammonia diffusion case because the local heat-transfer rate $q_w(x)$ was not measured directly and because the derivative of an empirically inferred length, namely l_x, always generates errors. All one can state definitely is that there are no clear inconsistencies between the Stanton number variation inferred by Johnson and that based on equation (12_t), where similarity is assumed (without the eddy-coefficient hypothesis).

† Johnson's identification of the similarity length with the convection thickness ([2b], Fig. 12), was a slip. In our Fig. 2, the similarity scale was worked our directly from original data and normalized so that 50 per cent of the T drop occurred in one scale length from the wall.

When the additional eddy coefficient hypothesis, equation (1), is made, the basic differential equation (6_t) takes the form

$$-\,U_e\, l_x\, d(x)\, \eta^{(n+1)/n}\left\{1 - \frac{\beta(x)}{n+1}\right\} g'(\xi) = [\epsilon_t(x, \xi)\, g'(\xi)]' \qquad (14_t)$$

The validity of the combined assumptions near the wall will be discussed later.

DEPENDENCE ON EDDY DIFFUSIVITY ON x

Since the right side of equation (14_C), divided by ϵ_C, is a pure function of ξ, strict similarity combined with eddy coefficient hypothesis would imply that ϵ_C must be separable into a product and must cancel out the x dependence on the left side. We find that ϵ_C can indeed be split, say into $\epsilon_{C1}(x)\ \epsilon_{C2}(\xi)$, so that

$$\frac{-\,U_e\, l l_x}{\epsilon_{C1}(x)}\left(\frac{l}{d}\right)^{1/n} \cdot \left(1 - \frac{\beta}{n+1}\right) = \left\{\frac{f'}{f}\right\}\frac{\epsilon_{C2}(\xi)}{\xi^{(n+1)/n}} \qquad (15_C)$$

Inasmuch as the split involves an arbitrary multiplicative constant, we may set both sides of equation (15_C) equal to -1, and proceed to obtain

$$\epsilon_{C1}(x) = U_e\, l l_x\left(\frac{l}{d}\right)^{1/n}\left(1 - \frac{\beta}{n+1}\right);$$

$$\epsilon_C(x, \xi) = \epsilon_{C1}(x)\, \epsilon_{C2}(\xi), \qquad (16_C)$$

$$\frac{C(x, y)}{C_w(x)} = f(\xi) = \exp\left\{-\int_0^\xi \frac{s^{(n+1)/n}}{\epsilon_{C2}(s)}\, \mathrm{d}s\right\}, \qquad (17_C)$$

Another degree of freedom remains in our definitions as can be easily verified from equation (17_C). Uniform stretching of the plume scale from $l(x)$ to $cl(x)$ changes the functions ϵ_C and f, but the solution remains similar and still satisfies the differential equations and the boundary conditions. Hereafter, we shall follow Poreh and Cermak and call $\lambda(x)$ that specific plume scale for which f is reduced to 0·5 when $y = \lambda$, i.e. $f(l) = 0{\cdot}5$.

Dimensional arguments convince us that $\epsilon_{C1}(x)$ must be proportional to the product of a characteristic velocity and characteristic length associated with the diffusive field. The natural characteristic length scale for the ammonia problem is $l(x)$, which in the intermediate and final zones specializes to $\lambda(x)$ and const $\cdot\,\delta(x)$ of Poreh and Cermak.† The candidates for the velocity scale are U_e, a constant, and $u_* = U_e\sqrt{(c_f/2)}$, a slowly varying function of x {roughly

$$\sim U_e^{n/n+1}\,\delta^{-1/(n-1)} \sim U_e^{n+2/n+3}\,(x+L)^{-1/n+3}$$

according to Hinze ([4], p. 476)}. In the intermediate zone, $\beta/(n+1)$ is still small in comparison with unity, and, to the same approximation, $d[\sim\delta(x)]$ and $\sqrt{(c_f/2)}$ are constant, so that equation (16c) leads to $\lambda \sim x^{n/n+1}$. The exponent $n/(n+1)$ has the values of 0·858 and 0·875 for $n = 6$ and 7, respectively, as compared to 0·8± observed experimentally by Poreh and Cermak. The choice of U_e for the velocity scale makes λ altogether independent of the ambient velocity, and the choice of u_* very nearly so, a probably significant agreement with the experiment. In the final zone, where $\beta \to 1$, equation (16c) yields $\lambda \sim x + \text{const.}$ and $\lambda \sim x^{n+1/n+2} + \text{const.}$, for the two choices of the velocity scale. These exponents are also somewhat higher than those which can be gleaned from Fig. 3 of [1]. Similarly, the results for both zones, combined with equations (9) and (12c), lead to a steady decline of the maximum concentration $C_w(x)$ some 5 to 10 per cent faster than the experimental values. Whether the cited relatively small discrepancies in the exponents indicate that a diffusion thickness more appropriate than λ (or δ in the final zone) is to be sought is not clear, but they appear unavoidable when the somewhat restrictive power law is used. Similar minor uncertainties exist in the interpretation of Johnson's experiments for step-heating, [2a], whether one uses a dimensional argument or the Reynolds analogy at the wall in the intermediate zone. There, the lack of direct measurement of q_w and c_f provides little opportunity to test the theoretical results for the x variations against experiment.

† We note that in the step-heating case, Eckert's convection thickness, $l(l/d)^{1n}\,i_t$, is also available, but it does not seem to offer any advantages. In the final zone the convection thickness is proportional to λ which is, in turn, proportional to δ. In these scalar transport cases, the equations are linear in C or T, and the variety of meaningful scales is less than for vortical layers (δ, δ^*, θ).

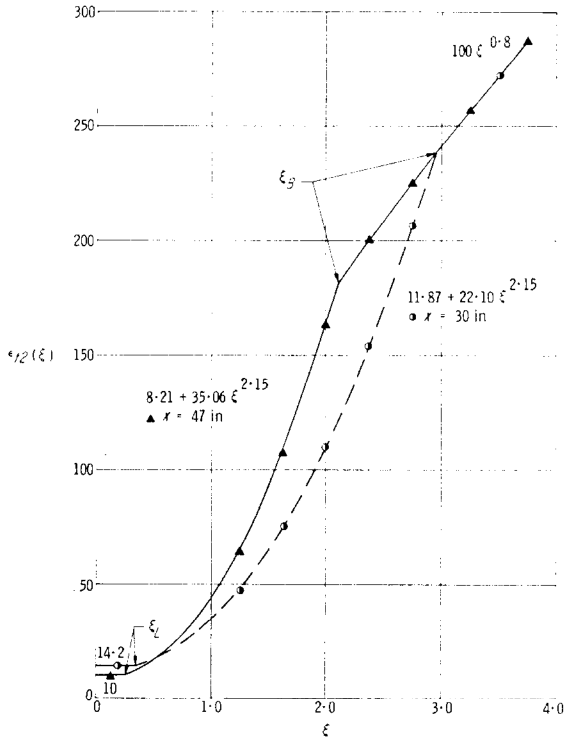

FIG. 3. Eddy conductivity variation near the wall.

In the step-heating case, equation (14t) leads us again to the separability of the eddy coefficient, $\epsilon_t(x, \xi) = \epsilon_{t1}(x)\,\epsilon_{t2}(\xi)$, so that the equivalent of equation (15c) becomes

$$\frac{-U\,ll_x}{\epsilon_{t1}(x)}\left(\frac{l}{d}\right)^{1/n}\left(1-\frac{\beta}{n+1}\right) = \left\{\frac{[\epsilon_{t2}(\xi)\,g'(\xi)]'}{\epsilon_{t2}(\xi)\,g'(\xi)}\right\}\left(\frac{\epsilon_{t2}(\xi)}{\xi^{(n+1)/n}}\right) \qquad (15_t)$$

Comparison of the role of the terms in the braces in equations (15t) and (15c) shows us that mathematically, it is the lateral development of dimensionless heat flux, $\epsilon_{t2}(\xi)\,g'(\xi)$, which corresponds not to the lateral development of mass flux but to that of the dimensionless concentration $f(\xi)$ itself. This reflects the aforementioned Heaviside-like and Dirac-like characteristics of the two diffusion problems. Setting both sides of equation (15t) equal to -1 again, we obtain

$$\epsilon_{t1}(x) = U_e\, l l_x \left(\frac{l}{d}\right)^{1/n} \left(1 - \frac{\beta}{n+1}\right); \quad \epsilon_t(x, \xi) = \epsilon_{t1}(x)\, \epsilon_{t2}(\xi) \qquad (16_t)$$

$$\frac{T(x, y) - T_e}{\Delta T} = g(\xi) = Q \int_\xi^\infty \frac{d\xi}{\epsilon_{t2}(\xi)} \exp\left\{-\int_\infty^\xi \frac{s^{n+1/n}}{\epsilon_{t2}(s)} ds\right\};$$

$$Q = \left[\int_0^\infty \frac{d\xi}{\epsilon_{t2}(\xi)} \exp\left\{-\int_0^\xi \frac{s^{(n+1)/n}}{\epsilon_{t2}(s)} ds\right\}\right]^{-1} \qquad (17_t)$$

The plume scale can be standardized again to $\lambda(x)$ such that $g(l) = 0{\cdot}5$, Fig. 2. The quantity Q is proportional to the heat flux at the wall; in fact, utilizing the properties of the differential equation one can show the relation to equation (12_t) to be $Q = i_t(n+1)/n$. It is the non-zero value of Q and the consequent second integration in equation (17_t) which makes the step-heating case much more sensitive to the wall conditions and to the finer variations of the still undetermined function $\epsilon_{t2}(\xi)$ than the ammonia diffusion case.

The second approximation to the dimensionless longitudinal momentum per unit volume, i.e. to U/U_e, bears strong resemblance to the approximate similarity solution for the dimensionless increase in heat per unit volume, $g(\xi)$, in the final zone (where approximate validity of Reynolds analogy can be expected). This approximation represents a second step in a fast-converging iteration process, operating on the basic momentum equation (10). Specifically, we retain the first power-law approximation $H_1(\eta) = \eta^{(n+1)/n}\, n/(n+1)$ for the weighting undifferentiated factor on the left side of equation (10) and treat the H derivatives as pertaining to the unknown second approximation, setting $H_2^{**}(\eta) = h^*(\eta)$. The results are to be compared to the solutions for concentration and temperature:

$$\epsilon_{u1}(x) = U_e\, d_x\, d \frac{n}{n+1}; \quad \epsilon_u(x, \eta) = \epsilon_{u1}(x)\, \epsilon_{u2}(\eta) \qquad (16_u)$$

$$\left.\begin{aligned}
&\frac{\tau(x, y)}{\tau_w(x)} = \frac{\epsilon_{u1}(x)\, \epsilon_{u2}(\eta)\, h^*(\eta)}{\tau_w(x)} = \exp\left\{-\int_0^\eta \frac{s^{n+1/n}}{\epsilon_{u2}(s)} ds\right\};\\
&\frac{U(x, y)}{U_e} = h(\eta) = \tau_{2w} \int_0^\eta \frac{d\eta}{\epsilon_{u2}(\eta)} \cdot \exp\left\{-\int_0^\eta \frac{s^{n+1/n}}{\epsilon_{u2}(s)} ds\right\};\\
&\tau_{2w} = \left[\int_0^\infty \frac{d\eta}{\epsilon_{u2}(\eta)} \cdot \exp\left\{\int_0^\eta \frac{s^{(n+1)/n}}{\epsilon_{u2}(s)} ds\right\}\right]^{-1}
\end{aligned}\right\} \qquad (17_u)$$

The constant, τ_{2w}, analogous to Q, is proportional to the shear at the wall,

$$\tau_{2w} = \tau_w(x)/\epsilon_{w1}(x).$$

The second approximations equation (17_u) differs from the family of solutions of equation (10) of Clauser [6] which collapsed into a narrow quasi-similar band around the empirical defect-law profile, $(U_e - U)/u_*$ versus η for $\eta > 0{\cdot}15$ or so, and which led to the expression for eddy viscosity,† equation (2). Rather, the approximations of equation (17_u) represent a refinement of the spirit and results of the power-laws, utilizing a single characteristic length across the complete layer. As such, their validity is likely to be restricted to narrower ranges of the x variable, but they should be adequate for comparison with the diffusion profiles in the rather narrow intermediate zone of quasi-similarity.

DEPENDENCE ON EDDY DIFFUSIVITY ON y

The ultimate test of the suitability of eddy diffusivity is, of course, the degree to which it

† There is little doubt that Clauser's fit to the experimental curves could be improved by making ϵ_u depend on y in order to account for the intermittency. Since the intermittency curves appear to scale with the inertial characteristic length δ, this refinement would follow the philosophy of the original investigation.

predicts the observed concentration profiles. Specifically, are there rational choices of $\epsilon_{C2}(\xi)$ in equation (17_C) and of $\epsilon_{t2}(\xi)$ in equation (17_t) which yield acceptable approximations to the experimental profiles of Poreh and Cermak in Fig. (1a) and of Johnson in Fig. 2, respectively? Furthermore, do such choices bear a physically sensible relation to a choice of the eddy viscosity variation $\epsilon_{u2}(\eta)$ in equation (17_u), which would yield a satisfactory match to the velocity profiles?

These questions are answered in the affirmative and the lessons learned about the effects on the "eddy mixing intensity" arriving at the matching mean profiles are discussed in great detail on pp. 22–37 and in the Appendix of [13], available from the author. Briefly, one's first inclination to solve for $\epsilon_2(\xi)$ in terms of the profiles and their derivatives and to utilize the experimental data for the evaluation of $\epsilon_2(\xi)$ founders on excessive scatter of the results. Instead, one is forced to an inverse procedure of trial-and-error estimates of $\epsilon_2(\xi)$, to numerical evaluation of equations (4) (which smooths out the effect of errors in the estimates), and to comparisons of the result with experimental data. Fortunately, it turns out that the tedious procedure is sweetened by the consequent appreciation for the contributions of the various segments of the layer (e.g. the so-called laminar sublayer, the Karman buffer layer, and the outer intermittent layer) to the impedance they offer to the transport of the scalar property in question.

However, because of the presence of integrals across the whole layer, like that in Q of equation (17_t), the absolute values do not emerge until the completion of the computations. It was therefore a pleasant surprise, when the ϵ_u values, corresponding to the profiles in Fig. 5, turned out to check well with those of other investigators in Fig. 4. For computational simplicity five analytical segments were used: $\epsilon_2 = \alpha_1$ in the laminar sublayer; $\epsilon_2 = a + b\eta^{2+(1/n)}$ in the "buffer layer"; $\epsilon_2 = \eta^p/\kappa$ for η up to 0·2, corresponding roughly to the logarithmic regime; $\epsilon_2 = \beta^*$ for the inertially controlled layer and $\epsilon_2 = A/\eta^{2.85}$ for a crude approximation of the intermittency drop-off. Adequate matching of the velocity profiles in Fig. 5 required the utilization of all five regions in spite of the smoothing effect brought about by the integrations.

Johnson's temperature profiles were determined primarily by the transport conditions within $\eta < 0.2$, i.e. by wall effects, so that only the inner three segments were in order. In the belief that the turbulent velocity field determines the heights η at which the mode of the heat transfer changes, an attempt was made to fit the temperature profiles by keeping the same η boundaries for the three layers and the same analytical form (but not magnitudes) of the corresponding ϵ_2 variations as in the velocity case.

The results shown in Fig. 2 indicate that this intuitive assumption on the spatial variation of the mixing mechanism was successful. The relative amplitudes of the eddy coefficients, ϵ_{u2} and ϵ_{t2} of course, change, and with them changes the local Prandtl number. On this basis, local Prandtl-number values in excess of unity should be reached in the early stages of the intermediate zone.

In the process of the successive adjustment of the viscous sublayer thickness as appeared dictated by the requirement of the matching of the mean profiles, the puzzling absence of a viscous parameter was clarified. In Fig. 3, the eddy coefficients, as would be required within the present framework by the proper growth of the sublayer between stations $x = 30$ in and $x = 47$ in are shown. The integrated effect of the impedance $1/\epsilon_{t2}$ across the two inner layers [see equation (17_t)] turns out to be very nearly the same as can also be judged from the corresponding points in Fig. 2.

Thus, what a strict similarity would require to be a single function, $\epsilon_{t2}(\xi)$, appears instead as a family of functions varying within a band near the wall. The corresponding mean temperature profiles also fall into a narrow band, and so do the experimental profiles—*the band of quasi-similarity*.† This study thus leads to a better

† Such "narrow-band similarity" at last endows the rather vague term of quasi-similarity in the Introduction with a more concrete meaning. Mathematical justifications for approximate validity of solutions (17_C), (17_t) and (17_u) must rest on the relative smallness of the departures from strict similarity and of their rate of change with respect to x. The widely used concept of "local similarity" of laminar, especially hypersonic, boundary layers relies on similar arguments.

understanding of what similarity, inferred from empirical data, may mean in more general cases.

As pointed out, the ammonia concentration profiles were expected to display little sensitivity to the viscous layer near the wall on theoretical grounds and accounting for the sublayer and the buffer layer was indeed found unnecessary. The shift in the mean profiles due to the inclusion of these layers was less than the experimental scatter. The blockage at the wall, however, makes the $C(x, y)$ profiles extend readily past the height $\eta = 0{\cdot}2$ into the inertially controlled regime and lose strict similarity on that account. [The heights $\eta = 0{\cdot}2$ at the beginning and end of the intermediate zone are shown in Fig. 1(b).] Again, there is a band of quasi-similarity within which both the experimental and the theoretical profiles fall, with either β or λ/δ in the role of a weak parameter. The examples in Fig. 1(b) illustrate the theoretical shift of the profiles in the intermediate similarity zone from the fully wall-governed profile, to the profile where "inertial cut-off" sets in at $y = \lambda$, and to the extreme case of cut-off at $y = 0{\cdot}6\lambda$, designated by symbols $+$, ▲, and $\times$, respectively.

As the ammonia plume penetrates deeper into the boundary layer, and approaches the final similarity zone, the sensitivity of the diffusivity model to the conditions near the wall decreases still further so that the assumption of constant ϵ across the whole boundary layer suffices for good results [Fig. 1(a)]. Reasonable assumptions of intermittency cut-off (not shown) improve the match with experimental results somewhat.

FINAL RESULTS AND DISCUSSION

When we combine the information developed piecemeal in the preceding sections we arrive at the overall representation of the diffusivities:

$$\epsilon(x, \xi) = U_e\, \lambda\, \lambda_x \left(\frac{\lambda}{d}\right)^{1/n} \left(1 - \frac{\beta}{n+1}\right) \epsilon_2(\xi) \tag{20}$$

In particular, in the intermediate zone of similarity of Poreh–Cermak, characterized by the experimental data:† $0{\cdot}15 < \lambda/\delta < 0{\cdot}36$; $0{\cdot}08 < \beta < 0{\cdot}3$; $\lambda \sim x^{0{\cdot}8+}$ (independently of U_e); $C_w \sim x^{-0{\cdot}9+}$; the function ϵ_{C2} can be approximated by expressions such as $0{\cdot}96\,(y/\lambda)^{2/3}$ of Fig. 1(a) or $1{\cdot}03\,(y/\lambda)^{0{\cdot}75}$ of Fig. 1(b), with a slowly varying cut-off and plateau values, when appropriate. (Within the experimental band of similarity, the profiles corresponding to the truncated distribution of the diffusivity could also be approximated with slowly decreasing exponents p, e.g. as low as $0{\cdot}4$: $\epsilon_{C2} = 0{\cdot}83\,(y/\lambda)^{0{\cdot}4}$; $g(\xi) = \exp[-0{\cdot}693\ \xi^{1{\cdot}5}]$.) Physically, the source was located at an approximate distance $L = 340$ in downstream of the effective origin of the turbulent layer, and the intermediate zone extended to about 150 in downstream of the source.

The final zone, of Poreh and Cermak corresponded to $\lambda/\delta \approx 0{\cdot}64$; $\beta \approx 0{\cdot}8 - 1$; $C_w\delta =$ constant; and $\epsilon_{C2}(y/\lambda) = 0{\cdot}67$ except for a near-Gaussian correction factor for the effect of turbulence intermittency for $y/\lambda > 0{\cdot}9$. The ammonia source was located at an approximate distance $L = 125$ in downstream of the effective start of the turbulent layer, and the final zone was reached about 400 to 500 in downstream of the source.

The intermediate zone of similarity of Johnson extended from 25–30 in to 60–65 in downstream of the leading edge of the heated plate, which in turn was located at an approximate distance $L = 168$ in downstream of the effective origin of the turbulent boundary layer. It corresponded approximately to $0{\cdot}055 < \lambda/\theta < 0{\cdot}95$; $0{\cdot}12 < d \ln \theta / d \ln \lambda < 0{\cdot}27$; $\lambda \approx 0{\cdot}00205\, x^{0{\cdot}79}$; and to the diffusivity function ϵ_{t2} displayed in Fig. 3. In general, this function changes slowly with x as the ξ values corresponding to the edges of the laminar and buffer layers change slowly with the development of the plume.

An interesting feature is the sensitivity of the concentration and temperature profiles to the power p in $\epsilon_2 = \xi^p/\kappa$ in the wall-controlled turbulent region [shown for the former in Fig. 1(a) and Fig. 1(b)]. While for the temperature profiles the best fit occurred for $p = 0{\cdot}8$, values between 0·6 and 0·7 appear best for the concentration profiles (as against the classical Prandtl value of unity). The demonstrated differences in the sensitivity to the viscous wall conditions and in the values of p suggest that the effective transport of scalar properties in the same tur-

† The lower limits of λ/δ and β may be smaller; profiles at the indicated limits were already similar.

bulent layer may depend on the boundary conditions of the scalar field. In other words the eddy-conductivity in a low-speed boundary layer may depend weakly on whether there is heat transfer or not at the wall.

Equation (20), with $\lambda = d$ and $\beta = 1$, also approximates the eddy viscosity of the turbulent layer itself over relatively short distances over which the exponent n in the first approximation of the velocity profile can be considered a constant. The function $\epsilon_{u2}(\xi)$ which corresponds to the boundary layer of Johnson in the region of intermediate development of the heated plume comprises five analytical segments displayed in Fig. 4. The x development of the laminar and the buffer layers over this distance has little influence on the velocity profiles and most of the layer could be characterized well by Clauser's constant value of eddy viscosity (Fig. 4). This approximation to ϵ_{u2} has the character of a final-zone approximation.

Although equation (20) applies to both the intermediate and the final zones of similarity, a clear analytical and physical distinction between the two resulting approximations was made in the process of its derivation because the postulate of quasi-similarity is not valid in the intervening region. Thus, equation (20) should provide a good basis for comparison of the diffusivity values within the intermediate zone and a somewhat less reliable basis for comparison of diffusivity values in the intermediate and the final zones of the same plume. For such comparisons over limited x regions, rather general conclusions were reached in [13] in terms of the approximations $\lambda = \lambda_0 x^m$ and $d = d_0 (x + L)^r$, where the ratio r/m is generally expected to lie between 0·9 and 1. In particular, these approximations furnish a good physical measure for β, namely† $x/(x + L)$.

The evidence in [13] for expecting effective turbulent Prandtl and Schmidt numbers in excess of those encountered in fully mixed layers ($\beta = 1$), on the basis of postulates of intermediate zone quasi-similarity appeared rather conclusive, especially near the beginning of the zone. In absence of a dynamic effect of the heating or the mass transfer on the turbulent structure of the boundary layer, the "mixing motions" in the intermediate zone could not be distinguished from those in a boundary layer heated uniformly from the leading edge at the same Reynolds number. Hence mixing-length hypotheses would lead to identical Pr_t and to a contradiction with the experimental evidence. Analytically, the eddy-coefficient hypothesis, suggested by the property of quasi-similarity, provides a more consistent picture primarily because the consequent equations indicate that the scalar diffusivities vary longitudinally almost

† Reynolds *et al.* [3] based their correlations on the parameter $1 - \beta = L/(L + x)$ both in the regions of similarity as well as in the non-similar initial and transitional regions.

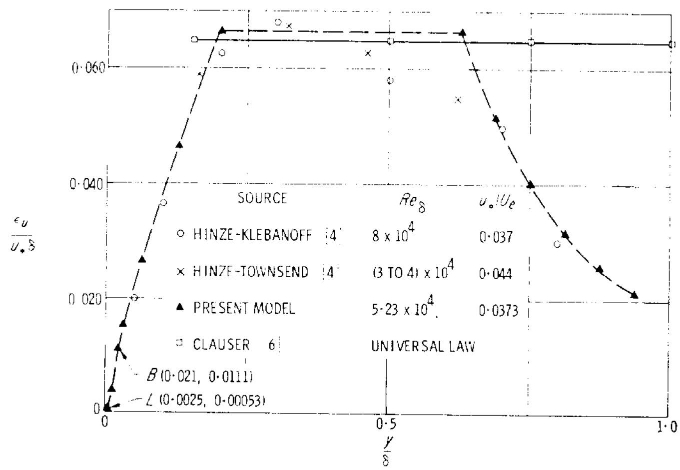

FIG. 4. Eddy viscosity variation across the boundary layer.

as the first power of the growing scale of the plume, $\lambda(x)$, not unlike Prandtl's second hypothesis for ϵ in free flows. Since the experimentally inferred lateral variations of the eddy conductivity and of the eddy viscosity were the same, it would appear that, at a fixed y (or y/δ) in the plume, the correlation $\overline{\tau v}$ is decreasing slower with x than the mean temperature gradient, $\partial T/\partial y$.

As mentioned at the outset, the sharpness of the thermal intermittency observed by Johnson testifies to a role of the large eddies in the transport mechanism. In fact, the height at which the intermittency of 50 per cent occurs is a good measure of the net lateral travel within the largest eddies carrying the heat signature in the convection time available since the passage by the station $x = 0$. At $x = 47$ in, i.e. in excess of 12 boundary-layer thicknesses δ downstream of $x = 0$, this height is 0·41 δ. According to the combined but still incomplete evidence at Favre *et al.* [14], Grant [15], Willmarth and Wooldridge [16], and Kline and Runstadler [17], a typical large eddy with an x-scale on the order of δ would decay in one-half to one-third the convective distance of 12 δ. (Both lateral motions of such eddies have considerably finer scales, especially near the wall.) Even if the transport were effected primarily by these large eddies, it would require a co-operative action of several successive and spanwise adjacent eddies. Actually, the net transport to $y = 0{\cdot}41\ \delta = 40\ \lambda$ was very small, resulting in a temperature increment of $\frac{1}{50}$ of total ΔT. Consequently, the significant transport in the region which determined the basic shape of the temperature profile (and hence of ϵ_2), $y < 25\ \lambda$, must have been amply modulated by the profuse hierarchy of smaller eddies of all sizes.

In fact, from Klebanoff's u spectra (e.g. Fig. 7–23 of Hinze [4]), it is known that the longitudinal scale of the eddies reaching below $y < 0{\cdot}05\ \delta$ steadily decreases. Furthermore, Grant [15] in his more extensive correlation studies finds evidence of a different complex of small eddies near the wall, $y < 0{\cdot}15\ \delta$ or so. Indeed, Johnson's comments on the finer scale ("burst") of the warmer outward moving fluid and the lesser perturbation of the relatively larger-scale (cool) v fluctuations, penetrating into the warm plume, could be interpreted as partial corroboration of Grant's inference of narrow "outward mixing

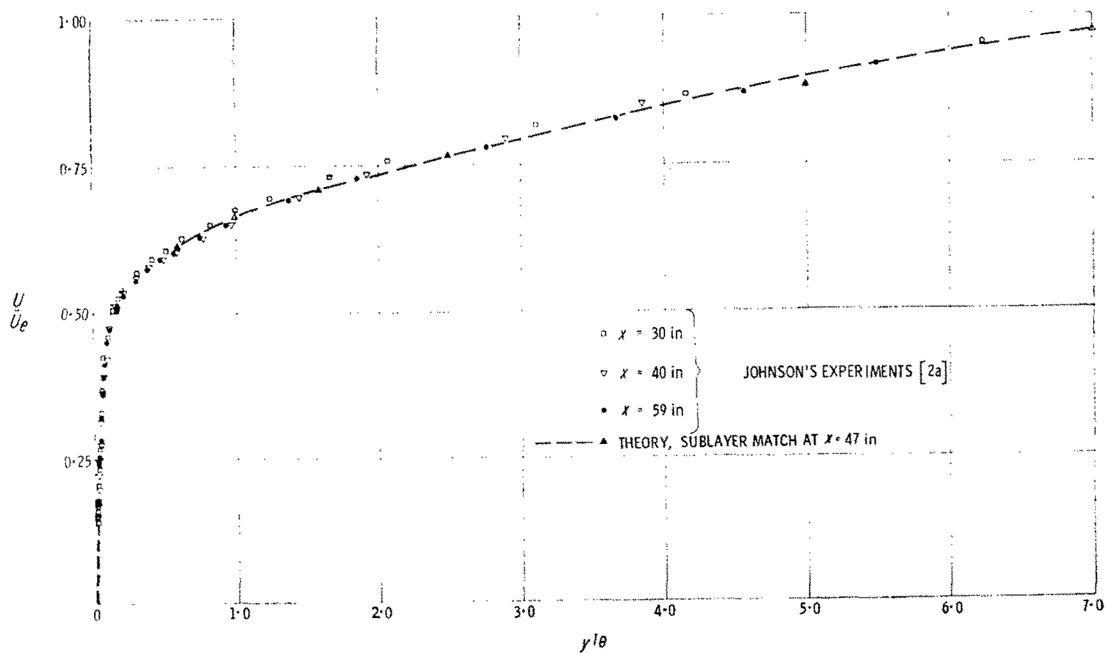

FIG. 5. Comparison of experimental and computed velocity profiles.

jets", and Lilley's [18] additional inference of larger-scale, somewhat less turbulent "return flow". These various observations seem to indicate that the early hot-wire experience with free turbulent flows may have led to an exaggerated anticipation of the importance of transport by the largest eddies in a turbulent layer in presence of a wall. While these eddies undoubtedly contribute, apparently ample smoothing action and randomization on finer scales takes place. It is also clear that further experiments on the structure of the turbulent boundary layer are needed and that configurations with plumes of scalar tracers offer special promise.

CONCLUSIONS

The experimental evidence of Poreh and Cermak [1], and that of Johnson [2a], both investigations exhibiting quasi-similarities in diffusive plumes, have been examined rather critically to see whether these quasi-similarities have a counterpart within the framework of a model of the turbulent fields utilizing the concept of eddy diffusivities. The following statements appear to be justified.

1. The ammonia diffusion experiments of Poreh and Cermak [1], specifically the observed shape of quasi-similar concentration profiles in the intermediate and final zones, are consistent with the concept of eddy diffusivity, viewed as a property of quasi-similar fields. Johnson's temperature profiles [2a] are similarly consistent. The nature of these quasi-similarities in presence of viscous and inertial scales, in addition to the plume scale, λ, has been clarified: as the additional scales vary with respect to λ, the profiles can be made to collapse into a narrow band because of a number of compensating effects.

2. When the velocity profiles are simulated locally by power profiles, this concept of eddy diffusivity leads to the experimentally observed result that in the first approximation the ammonia plume width $\lambda(x)$ grows independently of the free stream velocity U_e. The exponent of the predicted variation for λ, $x^{0\cdot86}$ to $x^{0\cdot875}$, is somewhat higher than that observed experimentally, namely $x^{0\cdot8}$.

3. The double quasi-similarity (in the intermediate and final zones) appears to be brought about by the gradual development of the scalar field in regions of different turbulent properties. Because of the impermeability of the wall, the concentration profiles are less sensitive to the direct viscous effects of the wall. However, in the intermediate zone the plume extends readily past the region near $y/\delta = 0\cdot2$ where the analytical character of the diffusivity changes rather abruptly from an approximate ξ^p variation ($0\cdot6 < p < 0\cdot8$). This causes the profiles to shift and pivot within a narrow band of quasi-similarity. The profile characteristics in the final zone are governed essentially by a constant diffusivity in the core of the layer and a smooth intermittency cut-off. The behavior resembles that of shear–stress variations in a fully developed vortical layer.

4. Johnson's temperature profiles are essentially dominated by wall effects which require a laminar, a buffer, and a ξ^p ($p \sim 0\cdot8$) region for adequate analytical simulation. The quasi-similarity with respect to y/λ can be achieved in a limited (intermediate) range of plume development where an effective lateral heat impedance, approximated by $1/\epsilon_{t2}(\xi)$, remains constant with x on the average.

5. The diffusivity coefficients, when viewed as a property of quasi-similar fields rather than of the medium, depend upon the local characteristic scales of the diffusing fields, which change during the course of their development even in presence of more or less invariant eddy structure of the surrounding turbulent field. It follows that the so-called turbulent Prandtl or Schmidt numbers are not absolute numbers, but depend upon the relative development of the turbulent velocity field and the scalar diffusing field. The theoretically inferred trends are in fair agreement with the Prandtl number values, locally in excess of unity, measured by Johnson [2] with a hot wire.

6. The present study (which included partial consideration of the initial zone) demonstrated to the author that postulates of eddy diffusivity for scalar fields with a history of development different from that of the carrier turbulent field may lead to illusory engineering results unless supported by additional information concerning the behavior of these fields while mixing-length hypotheses fail altogether. On the other hand, evidence such as that of the strong defect law,

utilized by Clauser [6] for prediction of boundary-layer characteristics over widest ranges of Reynolds number, and that of the present empirical narrow-band quasi-similarity in the intermediate zone of step-like plumes, can form a basis of fruitful eddy-diffusivity models. Their interpretation in terms of the more recent results in turbulence, advocated by Liepmann [10], not only lends them more support, explains and partially defines departures from stricter applicability, but in the present case also provides new ground for conjectures on the respective roles of small and large eddies in turbulent transport processes.

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Résumé—La preuve expérimentale par Poreh et Cermak [1] de l'existence de deux zones de quasi-similitude dans une nappe diffusant à partir d'une source linéaire à la paroi d'une couche turbulente établie est examinée avec soin pour la rendre compatible avec les concepts de diffusivité turbulente. La quasi-similitude des profils de température moyenne dans une couche turbulente établie, quelque distance en aval d'un saut de température pariétale, observée par Johnson [2a], [2c], est analysée de la même façon. On conclut que les diffusivités turbulentes considérées comme des propriétés de champs en quasi-similitude peuvent rendre compte des caractéristiques observées de la couche se développant dans une autre couche aux erreurs expérimentales près. Quelques conséquences de ces notions sont examinées.

Zusammenfassung—Der experimentelle Nachweis von Poreh und Cermak [1] für zwei Bereiche von "Quasiähnlichkeit" in einer Feder, die von einer linearen Quelle an der Wand einer ausgebildeten Grenzschicht wegwandert, wird sorgfältig auf einen Zusammenhang mit den Vorstellungen von einem turbulenten Austausch überprüft. Die von Johnson [2a], [2c] beobachtete Quasiähnlichkeit einer ausgebildeten turbulenten Schicht etwas unterhalb einer Stelle mit sprunghafter Änderung der Wandtemperatur wird genauso analysiert. Man kommt zu dem Schluss, dass ein turbulenter Stoffaustausch, wenn er als Eigenwert von quasi-ähnlichen Feldern betrachtet wird, die beobachteten Charakteristiken einer Grenzschicht, die in einer anderen Grenzschicht entsteht, bei der vorliegenden Beobachtungsgenaugikeit erklären kann. Einige Folgerungen aus diesen Zusammenhängen werden untersucht.

Аннотация—Экспериментально установленное Пори и Чермаком [1] существование двух зон квазиподобия в струе, диффундирующей от линейного источника на стенке в развитом турбулентном слое, тщательно рассмотрено с точки зрения совместимости с понятием о турбулентной диффузии.

Аналогично анализируется квазиподобие средних температурных профилей в развитом турбулентном слое на некотором расстоянии вниз по потоку от точки скачка температуры стенки, наблюдавшееся Джонсоном [2a], [2c]. Можно сделать вывод, что турбулентные диффузии, рассматриваемые как свойства квазиподобных полей, могут учитывать наблюдаемые характеристики слоя, развивающегося в другом слое, в пределах точности наблюдений. Исследованы некоторые следствия этих понятий.